$$\varphi(x, 0; t) = \frac{Ag^{1/2}t}{2\pi^{1/2}\rho_1 x^{3/2}} \sin\left(\frac{gt^2}{4x} + \frac{\pi}{4}\right)$$

Passing to the case $\gamma = \pi$, we note that although the function $\Phi_*(p, \theta)$ is multi-valued, the same reasoning applies as in the case just analyzed since we can consider its single-valued branch. The integral (2.3) assumes the form

$$\Phi(u, 0) = -\frac{1}{4\pi^{3/2} i} \int_{c-i\infty}^{c+i\infty} \Gamma\left(\frac{p+1}{2}\right) \mathrm{ctg}^{p/2+2} \frac{p\pi}{2} u^{-p}\, dp$$

An analogous decomposition of the integral into two parts, the use of asymptotic expressions for the integrand function, the same substitution of variables, and the application of the saddle point method lead to the asymptotic expression

$$\varphi(x, 0; t) = \frac{Ag^{1/2}t}{2\pi^{1/2}\rho_1 x^{3/2}} \sin\left(\frac{gt^2}{4x} + \frac{\pi}{4} + \frac{2}{\pi}\right)$$

In the case $\gamma = 3\pi / 4$, analogous considerations yield the formula

$$\varphi(x, 0; t) \approx \frac{Ag^{1/2}t}{3\pi^{1/2}\rho_1 x^{3/2}} \cos\frac{gt^2}{4x}$$

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CONDITIONS FOR SHOCKLESS STATE OF THE VORTEX FLOW IN TWO-DIMENSIONAL LAVAL NOZZLES

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Conditions given in [1, 2] for the absence of shocks in the flow in the vicinity of the center of a nozzle for two-dimensional vortex-free flows of an ideal gas are generalized to the case of rotational flows. Both continuous flows and flows with shock waves are constructed.

1. We take the origin of a Cartesian system of coordinates at the nozzle center, with the x-axis directed along the axis of the nozzle and the y-axis perpendicular to it. We assume that in the neighborhood of the nozzle center the entropy $S(y)$ is a sufficiently smooth function, so that the representation

$$S = s_0 y + {}^1/_2 s y^2 + o(y^2)$$

is valid.

Restricting ourselves to flows symmetric with respect to the x-axis, we set $s_0 = 0$. Then in the transonic approximation the system of equations describing the isoenergetic flow of an ideal gas has the form [3]

$$-uu_x + v_y = 0, \qquad v_x - u_y = sy \tag{1.1}$$

Let us pose the following Cauchy problem for the system (1.1) in the region $y \geqslant 0$ On the axis of symmetry $y = 0$

$$u = ax,\ x < 0; \qquad u = cx,\ x > 0 \tag{1.2}$$

$$v = 0 \qquad (a \geqslant c \geqslant 0)$$

We represent the required solution in self-similar form [4, 5]

$$u = |s|\, y^2 f(\xi), \qquad v = |s|^{3/2} y^3 g(\xi) \tag{1.3}$$

$$\xi = |s|^{-1/2} x y^{-2} \qquad (s \neq 0)$$

Substituting (1.3) into the system (1.1), we obtain two ordinary differential equations for f and g. Eliminating g from these equations, we obtain a nonhomogeneous second order equation for determining f

$$(f - 4\xi^2) f'' + (f')^2 + 2\xi f' - 2f = \text{sign}\, s \tag{1.4}$$

The function g is obtained from the relation

$$g = {}^1/_3\, [4\xi f + 2\xi\, \text{sign}\, s + (f - 4\xi^2) f'] \tag{1.5}$$

Equation (1.4) admits the simple particular solution [3]

$$f = A\xi + {}^1/_2\, (A^2 - \text{sign}\, s) \tag{1.6}$$

where A is an arbitrary constant. The single-parameter family of lines (1.6) has the envelope $f = -{}^1/_2\, (\xi^2 + \text{sign}\, s)$. Points ξ_0 of the parabola $f = 4\xi^2$ are singular points for the Eq. (1.4) and correspond to incoming and departing limiting characteristics, C_-° ($\xi_0 < 0$) and C_+° ($\xi_0 > 0$), respectively. For $A < 2\sqrt{2}/3$, sign $s = 1$, the line (1.6) does not intersect the parabola $f = 4\xi^2$. This means that the problem (1.2) has a solution only providing $a = c$, and the corresponding solution is analytic. To construct the solution for $A > 2\sqrt{2}\,/\,3$, sign $s = 1$, we divide the flow region into three parts: Region *1*, to the left of the characteristic C_-°; Region *2*, between the characteristics $C_\pm^\circ$; Region *3*, to the right of the characteristic C_+°. Using the solution (1.6), we can satisfy the initial conditions (1.2) by setting $A = a\,|s|^{-1/2}$ in Region *1* and $A = c\,|s|^{-1/2}$ in Region *3*. In the intermediate Region *2* the solution of Eq. (1.4) is obtained by a numerical integration from the continuity condition on the limiting characteristics.

Using Eq. (1.6), we can construct a continuous flow with weak discontinuities on the limiting characteristics. In Region *1* let the flow be described by the solution (1.6) with the constant A, and in Region *2* by the same solution with the constant B. By effecting a coalescence of these solutions on the characteristic C_-°, we obtain, in particular, $B = [(9A^2 - 8\ \text{sign}\, s)^{1/2} - 5A]\,/\,4$. The condition of continuity on the characteristic $C_+{}^\circ$ gives the value of the constant

$$C = -[(9B^2 - 8 \operatorname{sign} s)^{1/2} + 5B] / 4$$

in Region *3*. Analogously to the vortex-free case [2], the solution constructed is analytic and limiting in the sense that all the other integral curves which describe shockless gas flow are situated between them. The range of values of c / a corresponding to the continuous flows is determined by the inequality

$$\Phi_1(A) \leqslant c / a \leqslant 1, \qquad \Phi_1(A) = C / A$$

which generalize Frankl's inequalities to rotational flows. For $A \to \infty$ which is equivalent to $s \to 0$, we have $\Phi_1 \to {}^1/_4$. The graph of the function Φ_1 is shown in Fig. 1. Numerical analysis shows that for values of A belonging to the interval

$$2\sqrt{2} / 3 \leqslant A \leqslant 5\sqrt{2} / 6$$

no continuous solutions exist in Region *2* with the exception of the solution (1.6) with $B = A$. For $A < {}^1/_3$, sign $s = -1$, the weak discontinuity arriving at the nozzle center along $C_-{}^\circ$ gives the effect of an acceleration of the flow, an occurrence which is not possible in the vortex-free case.

Figure 2 presents the family of integral curves of Eq. (1.4) for $A = 0.1$, sign $s = -1$; to the Curves *1* and *2* there correspond $f = 4\xi^2$ and $f = -{}^1/_2(\xi^2 - 1)$.

2. In constructing the discontinuous solutions it is necessary to satisfy the boundary conditions at the front of the shock wave

$$f_2 + f_1 = 8\xi^2_2 \tag{2.1}$$

$$10\xi_2 f_2 + (f_2 - 4\xi^2_2) f_2' = 10\xi_2 f_1 + (f_1 - 4\xi_2^2) f_1'$$

The subscript 1 refers to the state of the gas ahead the shock wave, the subscript 2 — behind the shock wave. The coordinate ξ_2 of the shock front is yet to be defined. The relations (2.1) coincide with the corresponding conditions for vortex-free flows [2].

Figures 3 and 4 show the behavior of the integral curves for flows with shock waves. Points on the curves Γ_1 and Γ_2 correspond to the state of the gas ahead the shock wave and immediately behind it. The Curves *1* and *2* correspond to $f = 4\xi^2$ and $f = -{}^1/_2(\xi^2 + 1)$. In Fig. 3 ($A = 3$, sign $s = 1$) the integral curves considered are located below the limiting solution with weak discontinuities. Since $\xi_2 > 0$, the shock wave originates at the nozzle center and extends downstream. The velocity behind the shock can be either supersonic or subsonic. We note that extension of the flows in the Region *3* by the introduction of a shock wave cannot be accomplished uniquely; the corresponding integral curves may intersect Γ_1 in several points.

In Fig. 4 ($A = 1.5$, sign $s = 1$) the integral curves lying above the line (1.6) are physically unreal because of the limiting curves which are unavoidable by the introduction of a shock wave. This gives an indentation in the region $P_\pm$ of the existence of flows with a shock wave in Fig. 1. The remaining integral curves, describing flows with shock waves, are located in the region of existence of continuous flows. Therefore the prior appearance of a limiting curve in the flow is not a necessary condition for the formation of a shock wave, as is the case for vortex-free flows [2]. The integral curves may pass through the points Γ_1 either in a continuous manner or may undergo discontinuities. The coordinate of the shock front ξ_2 can be positive or negative, i.e. either departing or arriving shock waves are allowed.

In the range of values

$$0 < c / a < \Phi_1 (A)$$

a shock wave is formed and the flow velocity behind the shock increases in the direction towards the exhaust end of the nozzle.

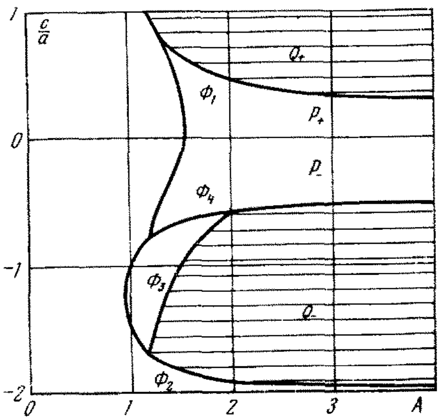

Fig. 1

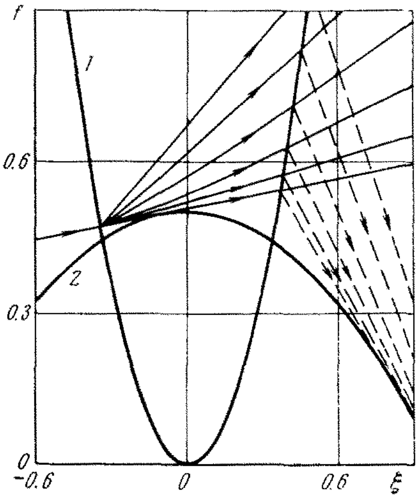

Fig. 2

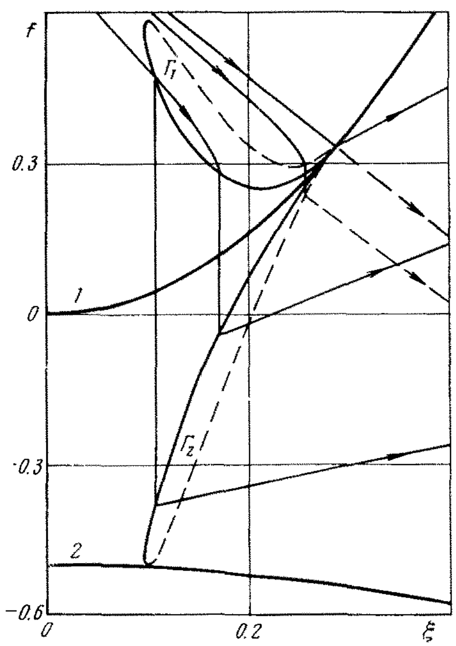

Fig. 3

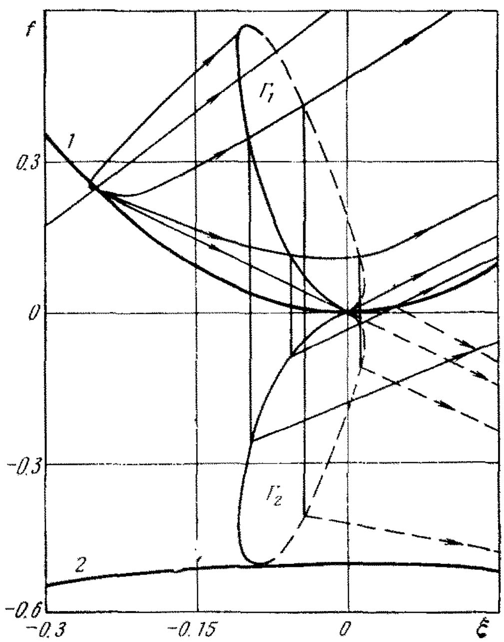

Fig. 4

3. Let us assume $a > 0$ and $c < 0$ in the Cauchy data (1.2). Then the values of c / a, defined by the inequalities

$$\Phi_2(A) \leqslant c / a \leqslant \Phi_3(A)$$

where

$$\Phi_2(A) = -[(9A^2 - 8 \operatorname{sign} s)^{1/2} + 5A] / 4A$$

$$\Phi_3(A) = \begin{cases} B / A, & A > 17\sqrt{2}/12 \\ -[(9C^2 - 8 \operatorname{sign} s)^{1/2} + 5C] / 4A, & A < 17\sqrt{2}/12 \end{cases}$$

correspond to the continuous flows.

For $A \to \infty$ $(s \to 0)$ we have $\Phi_2 \to -2$, $\Phi_3 \to -1/2$. If

$$\Phi_4(A) < c / a < 0, \qquad \Phi_4(A) = B / A.$$

then a shock wave arises and the flow velocity behind the shock decreases in the direction towards the exhaust end of the nozzle. The dependence of Φ_2, Φ_3, Φ_4 on A is shown in Fig. 1. Here the continuous flow corresponds to the region $Q_{\pm}$.

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ON STABILITY OF THREE-DIMENSIONAL PERIODIC MOTIONS IN HYDRODYNAMICS

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We obtain exact conditions for the stability of periodic motions. We show that the conditions found in [1] are necessary and sufficient, but they are only applicable to motions not dependent on time. The conditions given in [2] are applicable in the general case but are only sufficient (necessary) conditions of instability (stability). We consider the dependence of stationary motions on parameters.